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A via-point time optimization algorithm for complex sequential trajectory formation

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Abstract

In our previous research, we proposed a method for the reproduction of complex movement trajectories and robot arm control that could mimic fast, skilled human movements. That method is based on bi-directional theory and uses a representation of a set of via-points as boundary conditions or control variables to perform robot arm trajectory control. The via-points are extracted from human movement data and the resultant via-point representation is able to regenerate handwritten characters, control a Kendama toy, and perform a tennis serve. The via-point information contains both spatial and temporal information, that is, the position on the trajectory and the time of passing through the via-point position, respectively. Trajectory generation is performed using the trajectory formation model based on the optimal criterion, namely, the smoothness criterion, for which the boundary conditions are both the position and the timing of the via-point information. However, generating a smooth trajectory at different movement speeds is quite difficult if the time of passing through the via-point position is unknown or different from the extracted via-point time.

In this paper, we therefore propose a new algorithm which can determine temporal via-point information. Our proposed algorithm can generate roughly the same trajectory as the measured human trajectory from only the spatial information of via-point locations. The optimality and the convergence of the new algorithm are investigated theoretically, and the trajectory generated by the algorithm is shown in numerical experiments. It is shown that starting from arbitrary temporal information the proposed algorithm can produce an appropriate trajectory.

Keywords: Via-point; Time; Trajectory formation; Optimization

1. Introduction

In our previous research, we proposed a method for the reproduction of complex movement trajectories and robot arm control that could mimic fast, skilled human movements. That method is based on bi-directional theory (Kawato, 1995) and uses a representation of a set of via-points as boundary conditions or control variables to achieve robot arm trajectory control. The via-point representation is able to regenerate handwritten characters, control a Kendama toy, and perform a tennis serve (Miyamoto & Kawato, 1998; Miyamoto et al., 1996; Wada & Kawato, 1995).

The via-point representation we proposed is based on a computational theory for human arm movement control. Several computational theories have been proposed for human arm movement control and planning, including theories of the dependence on the dynamics of the controlled object, such as the minimum torque change criterion (Uno, Kawato, & Suzuki, 1989), minimum commanded torque change criterion (Nakano et al., 1999), minimum muscle tension change criterion (Dornay, Uno, Kawato, & Suzuki, 1996) and minimum motor command change criterion (Kawato, 1996). To generate a trajectory based on the above computational theories, a trajectory formation model based on the above criteria and the bi-directional approach (Kawato, 1995) has already been proposed, called Forward Inverse Relaxation Model (FIRM) (Wada & Kawato, 1993), which is shown to regenerate complicated human motion trajectories well. In the model, a complicated sequential trajectory is expressed using a set of via-points and the trajectory passing through the via-points

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is produced based on the minimum commanded torque change criterion.

Unfortunately, generating a slow movement trajectory and a fast movement trajectory for the same motion task is not possible using Wada and Kawato's trajectory formation model FIRM. This is because the via-point information contains spatial and temporal information, which are both boundary conditions, and this information is already learned and saved as position and time pairs. Therefore, in order to generate different movement speeds using FIRM, via-point information pairs for slow movement and for fast movement would need to be learned and saved in advance.

To overcome this drawback, in this paper, we propose a revised FIRM incorporating a new algorithm that can reproduce complicated trajectories without the need for a priori temporal information about the via-points. That is to say, our proposed new algorithm can determine the time of passing through the via-point position. It should be noted that our research aim here is not to reproduce a trajectory identical to the measured human trajectory, but to produce roughly the same trajectory from only the spatial information of the via-points. In the case of handwriting, it is important to be able to generate a script such as 'hello' that can be recognized by almost everyone, not to try to regenerate an 'ideal' script. Similarly, in the case of robot control, it is not necessary to teach the robot precisely in order for it to be able to execute a movement to succeed at a task.

The rest of our paper is as follows. In Section 2, we present a summary of FIRM based on the optimization principle, propose a new algorithm that can determine the time of passing through each via-point, and consider the theoretical foundations with respect to optimality and convergence. In Section 3, we present experimental results of trajectory formation by a two-joint manipulator. We conduct two experiments, the first of which is trajectory formation using one via-point and the second of which involves cursive handwritten character generation using several via-points, which is a complex sequential movement. We show that although the algorithm presented in our previous research cannot generate the appropriate trajectory without adequate temporal information the proposed algorithm can do so by using initial temporal information, which is given at random. Finally, in Section 4, we discuss the applicability of the proposed algorithm.

2. A computational trajectory formation model

2.1. A trajectory formation model based on the optimization principle

Fig. 1 shows the trajectory formation model, which consists of a hierarchical structure of motion planning (Wada & Kawato, 1995). Several conditions required for achieving movement are derived from visual information, for example, the start point, end point, and motion duration of a reaching movement are specified. These can also be regarded as a representation of the reaching movement. A minimization principle, namely, the minimum commanded

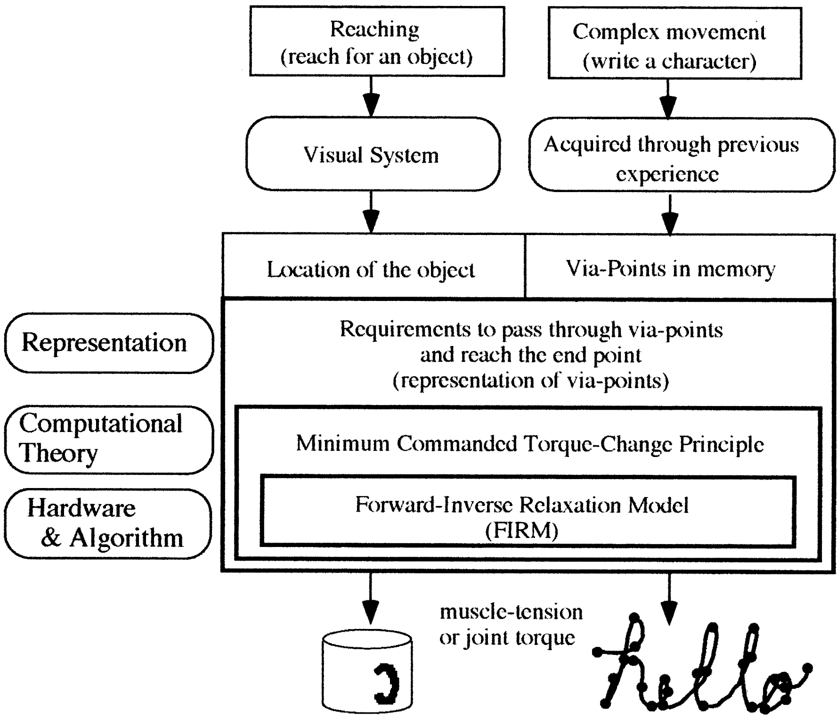

Fig. 1. Point-to-point movements and complex trajectory formation model.

torque change criterion (Nakano et al., 1999) is used to plan the trajectory. The minimum commanded torque change model is shown in the following equation:

$$C_{\mathrm{CTC}} = \frac{1}{2}\int_0^{t_f} \sum_{k=1}^{K} \left(\frac{\mathrm{d}\tau^k}{\mathrm{d}t}\right)^2 \mathrm{d}t \rightarrow \mathrm{Min}, \tag{1}$$

where τ^k is the commanded torque of joint k, K is the number of joints, and t_f is the motion duration. The FIRM (Wada & Kawato, 1993) provides a hardware model as well as an algorithm for generating the optimal trajectory. Finally, a joint torque or a muscle tension stream is computed in order to accomplish the movement and achieve the movement trajectory.

To plan a complex sequential movement, an optimization problem needs to be solved, which has boundary conditions that consist of many via-point positions (x, y and z coordinates) and the times of passing through those via-point positions. However, determining the via-point representation of a complex sequential trajectory is quite difficult. Wada and Kawato (1995) proposed an algorithm that can extract the via-points of a complex trajectory by solving an optimization problem as follows (Wada & Kawato, 1995). Let us assume that X_{data} is a given trajectory and X_{reconst} is a trajectory regenerated by the trajectory formation model. The via-point estimation problem is to find a set of minimum number of via-points n that satisfies Eq. (2):

$$|X_{\mathrm{data}} - X_{\mathrm{reconst}}| < \varepsilon. \tag{2}$$

That is, the via-point estimation algorithm finds the set containing the minimum number of via-points from among the sets of via-points that satisfy the error value of $|X_{\mathrm{data}} - X_{\mathrm{reconst}}|$ under the threshold ε. The extracted via-point information contains $(X_{\mathrm{via}}, t_{\mathrm{via}})$, where $X_{\mathrm{via}} = X_{\mathrm{data}}(t_{\mathrm{via}})$ ($\mathrm{via} = 1, 2, \ldots, n$). The coordinate X_{via} is the coordinate on X_{data} and the timing t_{via} shows the time of passing through X_{via}. Therefore, the trajectory formation model generates the trajectory X_{reconst} that passes through X_{via} at time t_{via}. The timing t_{via} is important information, along with the spatial coordinate X_{via}, for generating the trajectory.

FIRM consists of a Forward Dynamics Model (FDM) and an Inverse Dynamics Model (IDM). FIRM also has a mechanism for smoothly updating the torque command and a mechanism for generating the compensatory trajectory needed to satisfy the boundary conditions (Fig. 2). FIRM can generate a trajectory that has several via-points within a small number of iterations. The optimal trajectory based on the minimum commanded torque change model is obtained by repeating Steps 1 to 4 of Fig. 2. FIRM is an algorithm that solves the following optimization problem bound by both spatial and temporal information about the via-points:

$$C_{\mathrm{CTC}} = \frac{1}{2}\int_0^{t_f} \sum_{k=1}^{K} \left(\frac{\mathrm{d}\tau^k}{\mathrm{d}t}\right)^2 \mathrm{d}t \rightarrow \mathrm{Min} \tag{3}$$

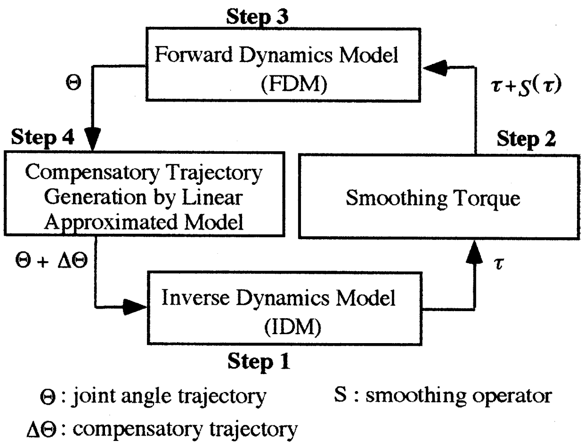

Fig. 2. Forward-Inverse Relaxation Model.

$$X_{\mathrm{reconst}}(0) = X_{\mathrm{start}},\quad \dot{X}_{\mathrm{reconst}}(0) = 0,\quad \ddot{X}_{\mathrm{reconst}}(0) = 0$$

(boundary conditions for start point)

$$X_{\mathrm{reconst}}(t_{\mathrm{via}}) = X_{\mathrm{via}}\ (\mathrm{via} = 1, 2, \ldots, n)$$

(boundary conditions for via-points)

$$X_{\mathrm{reconst}}(t_f) = X_{\mathrm{final}},\quad \dot{X}_{\mathrm{reconst}}(t_f) = 0,\quad \ddot{X}_{\mathrm{reconst}}(t_f) = 0$$

(boundary conditions for final point)

In the case of the trajectory formation with one via-point based on the minimum jerk criterion, the via-point time, t_1, is derived from the optimization techniques of Bryson and Ho (1975) and Flash and Hogan (1985):

$$\begin{aligned}
&\int_0^{t_f} \left(\frac{\mathrm{d}^3x}{\mathrm{d}t^3}\right)^2 + \left(\frac{\mathrm{d}^3y}{\mathrm{d}t^3}\right)^2 \mathrm{d}t \\
&\quad = \int_0^{t_1} \left(\frac{\mathrm{d}^3x}{\mathrm{d}t^3}\right)^2 + \left(\frac{\mathrm{d}^3y}{\mathrm{d}t^3}\right)^2 \mathrm{d}t \\
&\qquad + \int_{t_1}^{t_f} \left(\frac{\mathrm{d}^3x}{\mathrm{d}t^3}\right)^2 + \left(\frac{\mathrm{d}^3y}{\mathrm{d}t^3}\right)^2 \mathrm{d}t \rightarrow \mathrm{Min}
\end{aligned} \tag{4}$$

$$x(0) = x_{\mathrm{start}},\quad \dot{x}(0) = 0,\quad \ddot{x}(0) = 0$$

$$y(0) = y_{\mathrm{start}},\quad \dot{y}(0) = 0,\quad \ddot{y}(0) = 0$$

(boundary conditions for start point)

$$x(t_1) = x_1,\quad y(t_1) = y_1\quad (0 < t_1 < t_f)$$

(boundary conditions for via-point)

$$x(t_f) = x_{\mathrm{final}},\quad \dot{x}(t_f) = 0,\quad \ddot{x}(t_f) = 0$$

$$y(t_f) = y_{\mathrm{final}},\quad \dot{y}(t_f) = 0,\quad \ddot{y}(t_f) = 0$$

(boundary conditions for final point)

where x and y are hand position coordinates with respect to a Cartesian coordinate system and t_1 is not specified beforehand.

The analytical solution of the minimum jerk trajectory is given by a polynomial expression, however, that of the minimum commanded torque change trajectory cannot be derived. Therefore, in practice, it is impossible to determine more than 10 optimal via-point times in nonlinear optimization problems such as the minimum commanded torque change criterion.

2.2. A via-point time optimization algorithm

In the trajectory formation algorithm proposed in this paper, the time of passing through each via-point is not constrained. Rather, only the spatial information, that is, the position(s) of the via-point(s), is given as boundaries. Of course, the motion duration from the start point to the end point is specified. The computational framework of the trajectory formation model is same as the model shown in Fig. 1. That is, the via-point time optimization algorithm proposed in this paper is based on the minimum commanded torque change shown in Eq. (1).

However, two additional steps, Step 5 (minimization of the commanded torque change) and Step 6 (compensation for the motion duration), are added to FIRM, as shown in Fig. 3. We now explain the optimization algorithm of the via-point time shown in Fig. 3 in three parts.

(I) A trajectory is generated by applying FIRM to a set of initial via-point times.

(II) Next, in order to reduce the performance index of the minimum commanded torque change using the steepest descent method, the movement time between each via-point is updated in Step 5. The minimum commanded torque change criterion basically decreases when the movement time is lengthened (see Appendix A). The performance index of the minimum commanded torque change between via-point i and via-point $i-1$ is

$$C(t_i) = \int_0^{t_i} \sum_{k=1}^{K} \left(\frac{\mathrm{d}\tau^k}{\mathrm{d}t} \right)^2 \mathrm{d}t. \tag{5}$$

Here, t_i indicates the movement time between via-point $i-1$ and via-point i ($i = 1, 2, \ldots, n$). Via-point 0 represents the starting point and via-point n represents the end point.

The movement time between via-point $i-1$ and via-point i is updated with the following equation. This equation is derived from the steepest descent method:

$$\Delta t_i = \varepsilon \frac{1}{t_i} \int_0^{t_i} \sum_{k=1}^{K} \left(\frac{\mathrm{d}\tau_i^{*k}}{\mathrm{d}t} \right)^2 \mathrm{d}t, \tag{6}$$

where τ^* represents the commanded torque by which $C(t_i)$ is minimized and ε is an appropriate positive coefficient. Therefore, the movement time between the via-points shown above is extended according to the commanded torque change criterion and the entire motion duration exceeds the given motion duration, as shown in Fig. 4(A) and (B), respectively.

(III) In order to satisfy the given entire motion duration, the via-point time obtained in Step 5 is corrected in Step 6 according to the following equation (resulting in Fig. 4(C)):

$$t_i \leftarrow \frac{t_i + \Delta t_i}{t_\mathrm{f} + \Delta t_\mathrm{f}} t_f, \tag{7}$$

where $\Delta t_\mathrm{f} = \sum \Delta t_i$.

Thus, the proposed model includes two additional processes, which are similar to Step 2 and Step 4 in FIRM, namely, smoothing the commanded torque

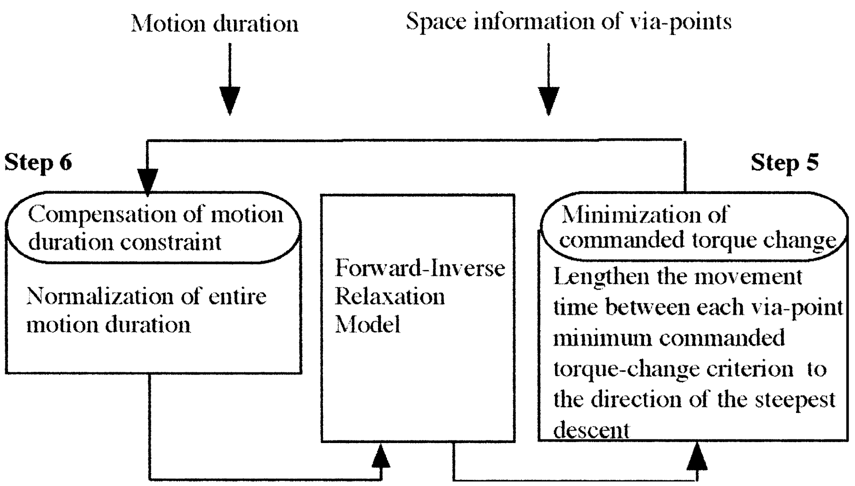

Fig. 3. Structure of via-point time estimation model.

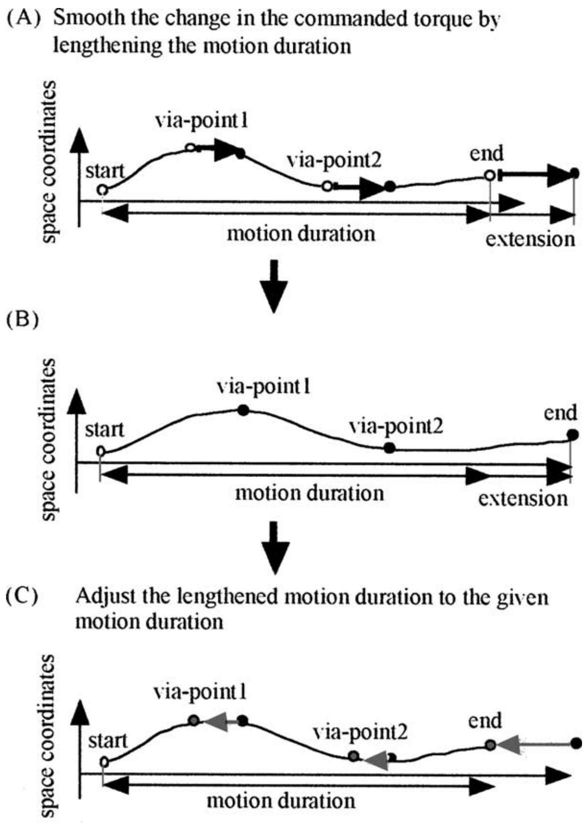

Fig. 4. Algorithm of via-point time estimation.

and compensating for error, respectively. The commanded torque is smoothed by lengthening the movement time using Eq. (6). The error of the motion duration is compensated by normalization using Eq. (7).

2.3. Theoretical considerations

In this section, the optimality and convergence of the new algorithm proposed in Section 2.2 are discussed. When the movement time, t_i, between via-points is given, we assume that the optimal commanded torque that satisfies the minimum commanded torque change criterion can be obtained. This means that the optimal trajectory of the minimum commanded torque change could be calculated with FIRM using the via-point time. The via-point time is corrected by normalization in Step 6 of Fig. 3 and the optimal trajectory is generated in the original FIRM, before returning to Step 5.

Let us assume that τ_i^* is the optimal commanded torque and that the optimal object function is denoted as in Eq. (8). The object function of the trajectory for the entire motion duration t_f is divided by the integration interval of each movement time between the via-points as shown in Eq. (9).

$$C_i^*(t_i) = \int_0^{t_i} \sum_{k=1}^{K} \left(\frac{\mathrm{d}\tau_i^{*k}}{\mathrm{d}t} \right)^2 \mathrm{d}t \qquad (8)$$

$$\begin{aligned} &C^*(t_1, t_2, \ldots, t_n) \\ &\quad = \int_0^{t_1} \sum_{k=1}^{K} \left(\frac{\mathrm{d}\tau_1^{*k}}{\mathrm{d}t} \right)^2 \mathrm{d}t + \int_0^{t_2} \sum_{k=1}^{K} \left(\frac{\mathrm{d}\tau_2^{*k}}{\mathrm{d}t} \right)^2 \mathrm{d}t \\ &\qquad + \cdots + \int_0^{t_n} \sum_{k=1}^{K} \left(\frac{\mathrm{d}\tau_n^{*k}}{\mathrm{d}t} \right)^2 \mathrm{d}t \\ &\quad = \sum_{i=1}^{n} \int_0^{t_i} \sum_{k=1}^{K} \left(\frac{\mathrm{d}\tau_i^{*k}}{\mathrm{d}t} \right)^2 \mathrm{d}t = \sum_{i=1}^{n} C_i^*(t_i) \end{aligned} \qquad (9)$$

where

$$t_f = \sum_{i=1}^{n} t_i.$$

The motion time change between the via-points is given by the steepest descent method as

$$\Delta t_i = -\varepsilon \frac{\partial C_i^*}{\partial t_i} \quad (\varepsilon > 0). \qquad (10)$$

The following equation is now assumed:

$$t = t_i s \quad (0 \le t \le t_i) \qquad (11)$$

$$\tau(t) = \tau(t_i s) = \tilde{\tau}(s) \qquad (12)$$

$$\begin{aligned} \frac{\partial C_i^*}{\partial t_i} &= \frac{\partial}{\partial t_i} \int_0^{t_i} \sum_{k=1}^{K} \left(\frac{\mathrm{d}\tau_i^{*k}}{\mathrm{d}t} \right)^2 \mathrm{d}t \\ &= \frac{\partial}{\partial t_i} \int_0^{1} \sum_{k=1}^{K} \left(\frac{1}{t_i} \frac{\mathrm{d}\tilde{\tau}_i^{*k}}{\mathrm{d}s} \right)^2 t_i \, \mathrm{d}s \\ &= -\int_0^{1} \sum_{k=1}^{K} \left(\frac{1}{t_i} \frac{\mathrm{d}\tilde{\tau}_i^{*k}}{\mathrm{d}s} \right)^2 \mathrm{d}s \\ &= -\frac{1}{t_i} \int_0^{t_i} \sum_{k=1}^{K} \left(\frac{\mathrm{d}\tau_i^{*k}}{\mathrm{d}t} \right)^2 \mathrm{d}t \end{aligned} \qquad (13)$$

Thus, we obtain the following equation, which is the same as Eq. (6):

$$\Delta t_i = \varepsilon \frac{1}{t_i} \int_0^{t_i} \sum_{k=1}^{K} \left(\frac{\mathrm{d}\tau_i^{*k}}{\mathrm{d}t} \right)^2 \mathrm{d}t \quad (\varepsilon > 0). \qquad (14)$$

The amount of update for the movement time between via-points is thus given by Eq. (14). Since Δt_i is positive, the motion time between each via-point is changed so that all via-point intervals become long. Now we consider the optimality and the monotone convergence of the proposed algorithm.

(1) *Optimality*. The motion duration, t_i, is updated by Eq. (14) and the time at which the via-points are passed through is recalculated to satisfy the given entire motion duration, as shown above in Section 2.2 (III). Then, the evaluation function of the optimal trajectory can be given

as in Eq. (15):

$$C^*(t_1,t_2,\ldots,t_n) = \sum C_i^*\left(\frac{t_i+\Delta t_i}{t_f+\Delta t_f}t_f\right) = \sum C_i^*\left(t_i + \frac{t_f\Delta t_i - \Delta t_f t_i}{t_f+\Delta t_f}\right) \tag{15}$$

Eq. (16) is obtained using Taylor's expansion of Eq. (15):

$$\begin{aligned} C^*(t_1,t_2,\ldots,t_n) &\cong \sum C_i^*(t_i) + \sum \frac{t_f\Delta t_i - \Delta t_f t_i}{t_f+\Delta t_f}\frac{\partial C_i^*}{\partial t_i} \\ &= \sum C_i^*(t_i) + \frac{\varepsilon}{t_f+\Delta t_f} \\ &\quad \times\left(\sum t_i \sum \Delta t_i \frac{\partial C_i^*}{\partial t_i} - \sum \Delta t_i \sum t_i \frac{\partial C_i^*}{\partial t_i}\right) \end{aligned} \tag{16}$$

The expression inside the parentheses of Eq. (16) is arranged using Eq. (10) as follows:

(expression inside the parentheses (16))

$$\begin{aligned} &= \varepsilon\left(\sum \frac{\partial C_i^*}{\partial t_i}\sum t_i \frac{\partial C_i^*}{\partial t_i} - \sum t_i \sum\left(\frac{\partial C_i^*}{\partial t_i}\right)^2\right) \\ &= \varepsilon \sum_{i<j}\left(t_i\frac{\partial C_j^*}{\partial t_j} - t_j\frac{\partial C_i^*}{\partial t_i}\right)\left(\frac{\partial C_i^*}{\partial t_i} - \frac{\partial C_j^*}{\partial t_j}\right) \end{aligned} \tag{17}$$

Therefore, Eq. (15) converges if

(A) $\dfrac{\partial C_i^*}{\partial t_i} = \dfrac{\partial C_j^*}{\partial t_j}$ or (B) $\dfrac{1}{t_i}\dfrac{\partial C_i^*}{\partial t_i} = \dfrac{1}{t_j}\dfrac{\partial C_j^*}{\partial t_j}$ holds.

First, we examine condition (A). The problem of determining via-point times should be equivalent to the following optimization problem with constraints. That is, the object function to be minimized should be the sum of the evaluation function of the minimum commanded torque change criterion between via-points. The constraint condition is such that sum of the time between via-points must be equivalent to t_f:

$$C^*(t_1,t_2,\ldots,t_n) = \sum C^*(t_i) \to \text{Min}, \tag{18}$$

where $t_1 + t_2 + \cdots + t_n = t_f$. Therefore, using Lagrange's multiplier method, the above problem takes a minimum value when Eq. (19)

$$\frac{\partial C_i^*}{\partial t_i} = \frac{\partial C_j^*}{\partial t_j} \text{ holds.} \tag{19}$$

That is, $C^*(t_1,t_2,\ldots,t_n)$ converges when (A) holds, and $t_1, t_2, \ldots,$ and t_n become the optimal solutions.

Next, condition (B) is considered. We can show that $C^*(t_1,t_2,\ldots,t_n)$ is not an optimal solution when condition (B) holds. Let us assume that t_i is updated by incrementing $\alpha\Delta t_i$ ($\alpha > 0$). In order to adjust the entire movement time to t_f, a movement duration, t_j ($j \neq i$), between via-points must be in decrement with $\alpha\Delta t_i$. Therefore, the following two equations are obtained using Taylor's expansion of C_i^*, which is the motion duration $t_i + \alpha\Delta t_i$, and C_j^*, which is the motion duration $t_j - \alpha\Delta t_i$.

$$C_i^*(t_i + \alpha\Delta t_i) \cong C_i^*(t_i) + \alpha\Delta t_i \frac{\partial C_i^*}{\partial t_i} \tag{20}$$

$$C_j^*(t_j - \alpha\Delta t_i) \cong C_j^*(t_j) - \alpha\Delta t_i \frac{\partial C_j^*}{\partial t_j} \tag{21}$$

The sum of C_i^* and C_j^* is given by Eq. (22) using the conditions from (B):

$$\begin{aligned} &C_i^*(t_i + \alpha\Delta t_i) + C_j^*(t_j - \alpha\Delta t_i) \\ &\quad = C_i^*(t_i) + C_j^*(t_j) + \alpha\Delta t_i\left(\frac{\partial C_i^*}{\partial t_i} - \frac{\partial C_j^*}{\partial t_j}\right) \\ &\quad = C_i^*(t_i) + C_j^*(t_j) - \alpha\varepsilon\left(\frac{\partial C_i^*}{\partial t_i}\right)^2\left(1 - \frac{t_j}{t_i}\right) \end{aligned} \tag{22}$$

That is, Eq. (22) decreases at $t_i > t_j$.

Accordingly, when (B) holds, $C^*(t_1,t_2,\ldots,t_n)$ is not an optimal solution. However, the evaluation function converges when t_i is equivalent to t_j.

The object function converges and reaches the optimal value when the movement time average of the minimum commanded torque change criterion between each via-point is equal, that is, when Eq. (19) holds. When condition (B) holds, the algorithm can converge; however, this solution is not necessarily optimal. Therefore, the convergence of the algorithm is a necessary condition for an optimal solution, but it is not a sufficient condition.

(2) *Monotone decrease*. Here, we show that the performance index decreases as a result of the iterative calculations of Step 5 and Step 6 in Fig. 3. Eqs. (16) and (17) show that the following equations hold.

$$\begin{aligned} &C^*(t_1^{l+1}, t_2^{l+1}, \ldots, t_n^{l+1}) \\ &\quad = \sum C_i^*\left(\frac{t_i^l + \Delta t_i^l}{t_f + \Delta t_f^l}t_f\right) \\ &\quad \cong \sum C_i^*(t_i^l) + \sum \frac{t_f\Delta t_i^l - \Delta t_f^l t_i^l}{t_f + \Delta t_f^l}\frac{\partial C_i^*}{\partial t_i^l} \\ &\quad = \sum C_i^*(t_i^l) + \frac{1}{t_f + \Delta t_f^l}\left(\sum t_i^l \sum \Delta t_i^l \frac{\partial C_i^*}{\partial t_i^l} - \sum \Delta t_i^l \sum t_i^l \frac{\partial C_i^*}{\partial t_i^l}\right) \\ &\quad = \sum C_i^*(t_i^l) + \frac{\varepsilon}{t_f + \Delta t_f^l} \\ &\qquad \times\left\{\sum_{i<j}\left(t_i^l\frac{\partial C_j^*}{\partial t_j^l} - t_j^l\frac{\partial C_i^*}{\partial t_i^l}\right)\left(\frac{\partial C_i^*}{\partial t_i^l} - \frac{\partial C_j^*}{\partial t_j^l}\right)\right\} \end{aligned} \tag{23}$$

Here, t_i^l and Δt_i^l show the movement time and increments of movement time between via-point $i-1$ and via-point i, respectively. In addition, l shows the l-th iteration in the iterative calculation.

We assume that the commanded torque change is smoother when the movement duration is longer (see Appendix A). This assumption yields Eq. (24) for

$t_i > t_j$.

$$0 > \frac{\partial C_i^*}{\partial t_i} > \frac{\partial C_j^*}{\partial t_j} \tag{24}$$

Eq. (25) also holds at this time.

$$t_i \frac{\partial C_j^*}{\partial t_j} - t_j \frac{\partial C_i^*}{\partial t_i} < 0 \tag{25}$$

Therefore, the following equation is based on Eqs. (24) and (25):

$$C^*(t_1^{l+1}, t_2^{l+1}, \dots, t_n^{l+1}) - C^*(t_1^l, t_2^l, \dots, t_n^l)$$
$$= \frac{\varepsilon}{t_f + \Delta t_f^l} \left\{ \sum_{i<j} \left(t_i^l \frac{\partial C_j^*}{\partial t_j^l} - t_j^l \frac{\partial C_i^*}{\partial t_i^l} \right) \left(\frac{\partial C_i^*}{\partial t_i^l} - \frac{\partial C_j^*}{\partial t_j^l} \right) \right\} < 0 \tag{26}$$

The performance index C^* steadily decreases, as shown by Eq. (27).

$$C^*(t_1^{l+1}, t_2^{l+1}, \dots, t_n^{l+1}) < C^*(t_1^l, t_2^l, \dots, t_n^l) \tag{27}$$

Hence, the performance index uniformly converges as a result of the iterative calculation shown in Fig. 3.

3. Experimental results

The numerical results of the trajectory generation are shown as follows. The forward dynamics model and the inverse dynamics model of FIRM are manipulators of two joints shown in the equations below.

$$\tau_1 = (I_1 + I_2 + 2M_2L_1S_2\cos\theta_2 + M_2(L_1)^2)\ddot{\theta}_1$$
$$+ (I_2 + M_2L_1S_2\cos\theta_2)\ddot{\theta}_2 - M_2L_1S_2(2\dot{\theta}_1 + \dot{\theta}_2)\sin\theta_2$$
$$+ B_{11}\dot{\theta}_1 + B_{12}\dot{\theta}_2 \tag{28}$$

$$\tau_2 = (I_2 + M_2L_1S_2\cos\theta_2)\ddot{\theta}_1 + I_2\ddot{\theta}_2 + M_2L_1S_2(\dot{\theta}_2)^2$$
$$\times \sin\theta_2 + B_{21}\dot{\theta}_1 + B_{22}\dot{\theta}_2 \tag{29}$$

Here, τ_i, θ_i, $\dot{\theta}_i$ and $\ddot{\theta}_i$ represent the actuated torque, position, velocity, and acceleration of each joint, respectively. In addition, M_i, L_i, S_i, I_i, and B_{ij} represent the mass, length, distance from the mass center to the joint, the rotary inertia of link i around the joint, and the coefficients of viscosity, respectively. B_{ij} shows that the joint angle velocity of link j influences the actuated torque of link i. Joints 1 and 2 correspond to the shoulder and the elbow, respectively. Joint 1 is located at the origin of the coordinate system. The value of each parameter is shown in Table 1. Importantly, the coefficients of viscosity are assumed to be the values indicated by Nakano et al. (1999). The following parameter values are used in the experiments:

$$\varepsilon = \beta \frac{1}{\sum \Delta t_i^l}, \qquad \Delta t_i^l = \frac{1}{t_i} \int_0^{t_i} \sum_{k=1}^{K} \left(\frac{d\tau_i^{*k}}{dt} \right)^2 dt \qquad (\beta = 0.65)$$

(1) *Movement with one via-point*. Fig. 5 shows the result of generating the trajectory with one via-point (start point S, via-point V, and end point E). The trajectory was measured using a three-dimensional position measurement device (OPTOTRAK3020, Northern Digital Inc.). The sampling frequency was 100 Hz. The measured motion was restricted to the horizontal plane by two degrees of freedom for the shoulder and the elbow. The trajectory shown in Fig. 5 was measured from one subject.

The start point and the end point were determined by searching for a point below 5% of the peak velocity and were

Table 1
Parameters of dynamics

Parameter	Link 1	Link 2
M_i (kg)	1.4950	1.0600
L_i (m)	0.2750	0.3570
S_i (m)	0.1130	0.1600
I_i (kg m^2)	0.0294	0.0405
B_i (kg m^2/s)	0.7	0.8
B_{ij} (kg m^2/s)	0.18	0.18

The three-dimensional shape of a male's arm was measured by a Cyberware Laser Range Scanner (Koike & Kawato, 1995). We calculated the arm as a homogeneous material with a specific gravity of 1.0 and computed its mass, center of mass, and moment of inertia from its volume. The arm parameters for the subject were calculated using the ratio of the arm length based on the measured data. Viscosity B_{ij} was calculated by basically the same method as Nakano et al. (1999).

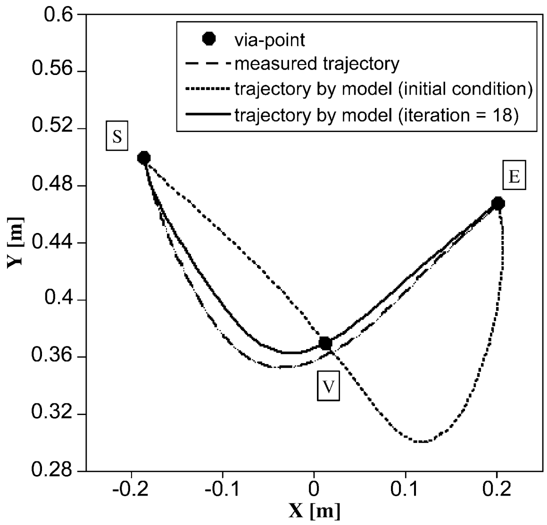

Fig. 5. Trajectory of via-point movement S–V–E (number of iterations = 18). The dotted line shows the trajectory generated by FIRM when the via-point time is 0.25 s. The solid line is the trajectory generated after 18 iterations. The dashed line corresponds to the measured trajectory.

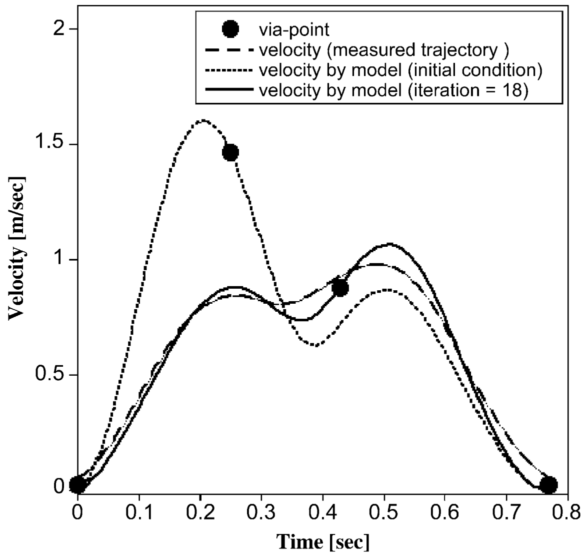

Fig. 6. Speed profile of S–V–E movement. The dotted line shows the speed profile generated by FIRM when the via-point time is 0.25 s. The solid line is the speed profile generated after 18 iterations. The dashed line is the velocity profile of the measured trajectory.

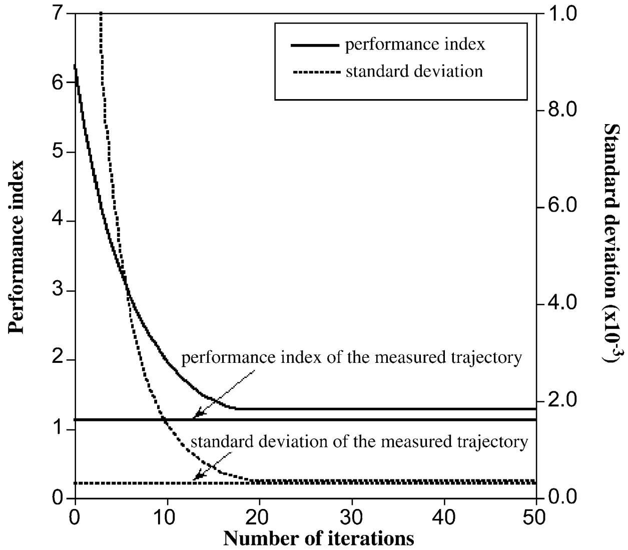

Fig. 7. Convergence of minimum commanded torque change criterion and the average time change of the criterion. The solid line shows the performance index of the minimum commanded torque change criterion. The performance index converges at between approximately 18 and 25 iterative calculations. The dotted line shows that the standard deviation of the average of the minimum commanded torque change criterion between via-points becomes small. Moreover, the performance index and the standard deviation of the measured trajectory are close to the values of the convergent points using the proposed algorithm.

used in the numerical experiment. For the via-point, a set point position was used. The duration of the motion was 0.77 s. An initial value of the via-point time was given as a random number; 0.25 s in this experiment. Fig. 6 shows the tangential velocity. For the trajectory produced using the via-point time of 0.25 s, the velocity shows a large fluctuation in the first half, while in the latter half the path is more curved than the measured trajectory. The reproduced trajectory is not smooth and is quite different from the measured trajectory. However, after 18 iterations, the via-point time is estimated as 0.43 s and both the path and velocity do not show such a large variation and are comparatively smooth. Both results come within a reasonable distance of the measured trajectory. Since the via-point V is located almost at the midpoint between the start point S and the end point E, the via-point time need only be approximated at the middle in the entire movement time 0.77 s. The velocity profile shows that the time of passing through the via-point is that for a point in the direction of the end point near the velocity minimum. This is one feature of the optimal trajectory that is generated by the minimum commanded torque change criterion.

Both the performance index of the minimum commanded torque change criterion and the standard deviation of the average (Δt_i^l) of the minimum commanded torque change criterion between via-points are plotted in Fig. 7. In this figure, the horizontal axis shows the number of iterative calculations. The minimum commanded torque change criterion decreases and converges through iterative calculations. Moreover, the standard deviation of the average of the minimum commanded torque change criterion between via-points also converges to a small value at the same time. This means that the average of the minimum commanded torque change between via-points becomes almost equal. Thus, the optimal via-point times can be obtained from the theoretical considerations described in Section 2.3. Also, the value of the minimum commanded torque change criterion and the standard deviation of the average (Δt_i^l) for the regenerated trajectory are approximately equivalent to the performance index and the standard deviation of the measured trajectory. The standard deviation of the average (Δt_i^l) of the minimum commanded torque change criterion between via-points for the measured trajectory is calculated using the via-point time (0.43 s) estimated by the via-point estimation algorithm (Wada & Kawato, 1995).

(2) *Movement with several via-points (handwritten characters 'abc')*. Next, as an example of complex trajectory formation, in Fig. 8 we show the results of handwritten movement. We used the measured trajectory shown in Fig. 8 in the same way as for Fig. 5. The via-point positions were extracted from a human handwriting trajectory for 'abc' (Wada & Kawato, 1995). Here, the via-point times were given as random numbers so as not to invert the order of the via-points. In this experiment, 15 sets of initial via-point times were tested. The average ± standard deviations of the given random via-point times were as follows: 0.09 ± 0.08, 0.23 ± 0.11, 0.35 ± 0.13, 0.49 ± 0.13, 0.65 ± 0.15, 0.81 ± 0.15, 0.96 ± 0.13, 1.10 ± 0.13, 1.27 ± 0.16, 1.46 ± 0.20, 1.63 ± 0.19, 1.80 ± 0.21, 2.00 ± 0.21, 2.19 ± 0.21, 2.33 ± 0.18, 2.48 ± 0.20, 2.69 ± 0.21, 2.88 ± 0.27, 3.06 ± 0.31, 3.32 ± 0.33, and 3.52 ± 0.34 s. The start point and end point times were 0.0 and 4.18 s, respectively. Then, we generated trajectories using the revised FIRM. The following

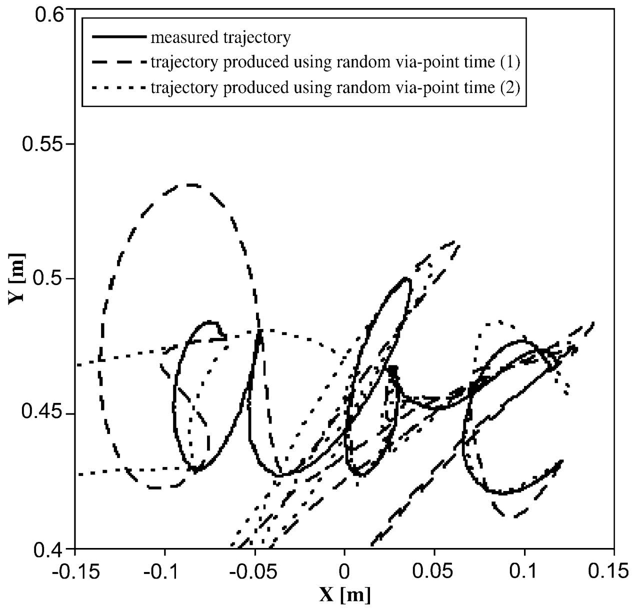

Fig. 8. Handwritten characters ‘abc’ (initial condition). The dotted line and the dashed line show the trajectories generated by FIRM using two of the initial via-point times. FIRM cannot regenerate the original human trajectory.

are the via-point times determined by starting from the above random initial via-point times: 0.22 ± 0.015, 0.56 ± 0.021, 0.76 ± 0.027, 1.16 ± 0.044, 1.37 ± 0.048, 1.50 ± 0.052, 1.69 ± 0.052, 1.83 ± 0.039, 2.11 ± 0.042, 2.27 ± 0.052, 2.41 ± 0.047, 2.56 ± 0.052, 2.68 ± 0.054, 2.85 ± 0.071, 2.99 ± 0.063, 3.15 ± 0.038, 3.30 ± 0.027, 3.42 ± 0.024, 3.54 ± 0.022, 3.74 ± 0.022, and 3.95 ± 0.026 s. The number of iterative calculations was 35. The via-point times estimated by the previous via-point estimation algorithm (Wada & Kawato, 1995) are 0.33, 0.58, 0.92, 1.32, 1.56, 1.69, 1.76, 1.86, 2.07, 2.30, 2.40, 2.48, 2.61, 2.70, 2.86, 2.99, 3.08, 3.16, 3.26, 3.58, and 3.81 s. The average absolute value of the difference between the random via-point time and the via-point time estimated by Wada and Kawato (1995), and the via-point time estimated by the proposed algorithm and the via-point time estimated by Wada and Kawato (1995) are 0.57 and 0.13 s, respectively. The random via-point times are modified toward the via-point time estimated by Wada and Kawato (1995). That is, the via-point time decided by the new algorithm seems to be appropriate.

The generated trajectory using random via-point times for the two sets shown in Fig. 8 is markedly different from the human handwriting data. Figs. 9 and 10 show the all regenerated trajectories. The trajectories shown in Fig. 9 approximate those of the human handwriting data. The tangential velocity profile is shown in Fig. 10. Though

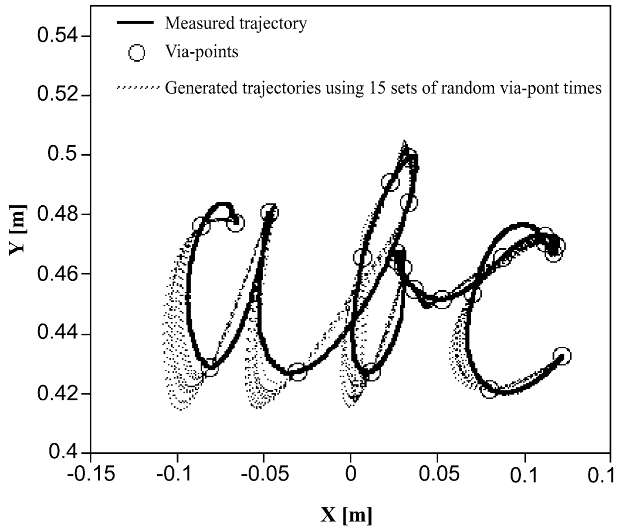

Fig. 9. The handwritten characters ‘abc’ (number of iterations = 35). The dotted line is the trajectory generated after 35 iterations.

the profile is slightly different from the actual data, the minimum or maximum points appear to be close to the actual movement.

Next, we examined the trajectory errors as follows:

(a) Error between the measured trajectory x^{measured}, y^{measured} and the trajectory $x^{\mathrm{originalFIRM}}$, $y^{\mathrm{originalFIRM}}$ produced using the via-point times that were estimated by Wada and Kawato’s (1995) model,

$$\mathrm{Err_a} = \sqrt{(x^{\mathrm{originalFIRM}}(t) - x^{\mathrm{measured}}(t))^2 + (y^{\mathrm{originalFIRM}}(t) - y^{\mathrm{measured}}(t))^2};$$

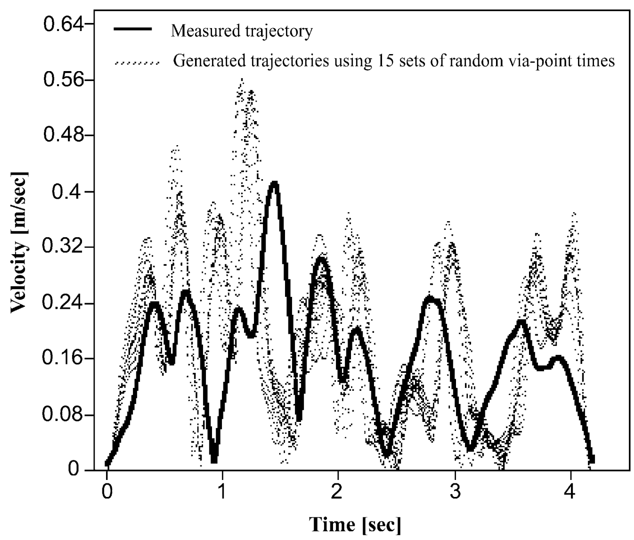

Fig. 10. Speed profile of the handwritten characters ‘abc’. The dotted line is the speed profile generated after 35 iterations.

(b) Error between the measured trajectory and 15 trajectories $x_i^{\text{revisedFIRM}}$, $y_i^{\text{revisedFIRM}}$ produced by the proposed model,

$$\mathrm{Err_b} = \frac{1}{15}\sum_{i=1}^{15}\sqrt{(x_i^{\text{revisedFIRM}}(t) - x^{\text{measured}}(t))^2 + (y_i^{\text{revisedFIRM}}(t) - y^{\text{measured}}(t))^2};$$

(c) Error between the measured trajectory and 15 trajectories x_i^{random}, y_i^{random} produced using initial random via-point times,

$$\mathrm{Err_c} = \frac{1}{15}\sum_{i=1}^{15}\sqrt{(x_i^{\text{random}}(t) - x^{\text{measured}}(t))^2 + (y_i^{\text{random}}(t) - y^{\text{measured}}(t))^2}.$$

The results were $\mathrm{Err_a} = 1.44$, $\mathrm{Err_b} = 11.0$, $\mathrm{Err_c} = 42.4$. Error (b) for our proposed algorithm is larger than error (a); however, it is shown that the proposed algorithm can generated almost the same trajectory as the measured trajectory without appropriate via-point times.

Moreover, Fig. 11 shows that both the performance index of the minimum commanded torque change criterion and the standard deviation of the average ($\Delta t_i'$) of the minimum commanded torque change criterion between the via-points decrease and converge. The value of the minimum commanded torque change criterion of the regenerated trajectory is approximately equivalent to the performance index of the measured trajectory. The value of the standard deviation of the average ($\Delta t_i'$) of the minimum commanded torque change criterion between the via-points of the regenerated trajectory is approximately equivalent to the standard deviation of the measured trajectory and also converges to a very small value. In other words, the proposed algorithm can be used to estimate optimal via-point times.

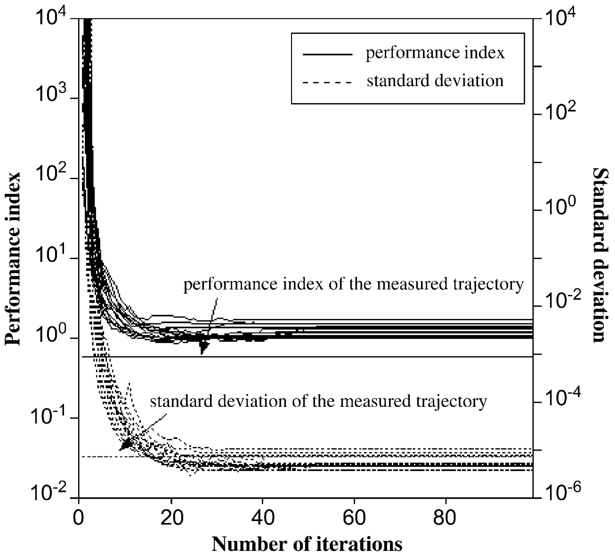

Fig. 11. Convergence of the minimum commanded torque change criterion and the average time change of the criterion. The solid line shows the performance index of the minimum commanded torque change criterion. The dotted line shows the standard deviation of the average of the minimum commanded torque change criterion between via-points. Both the decrease and convergence are similar to the movement of one via-point (Fig. 7). The performance index and standard deviation of the measured trajectory are similar to the values of the convergent points obtained using the proposed algorithm.

4. Discussion

In the present paper, we have proposed the incorporation of a new algorithm to the FIRM to estimate via-point time when the via-point time is not given in the motion plan and have clarified the theoretical background thereof. Moreover, the model's suitability was proved in a numerical experiment by applying the via-point time optimization algorithm to a trajectory formation with one via-point and to a more complicated trajectory formation containing several via-points.

Humans can write characters at different movement speeds once they have learned to do so at a certain speed without the need to learn to write at all those speeds. Our algorithm determines the via-point time(s) for various entire movement speeds from via-point information learned at just one particular speed. Fig. 12 shows that a faster movement (motion duration 2.09 s) can be adequately generated using the via-point spatial information of the movement with a motion duration of 4.18 s.

Modification of the via-point times according to intended motion duration is easy and is described below.

$$t_i^{\text{intended}} = t_i^{\text{learnt}} \frac{T^{\text{intended}}}{T^{\text{learnt}}},$$

where t_i^{intended} and t_i^{learnt} indicate the movement time between via-point $i-1$ and via-point i for the intended movement and learned movement, respectively. T^{intended} and T^{learnt} denote the intended motion duration and learned motion duration, respectively. In the experiments, we used random numbers as the initial via-point times. Here, t_i^{intended} can be used as appropriate initial values. The proposed algorithm produced, as expected, an appropriate trajectory in a short time if the learned motion duration and intended motion duration were similar. The proposed algorithm also

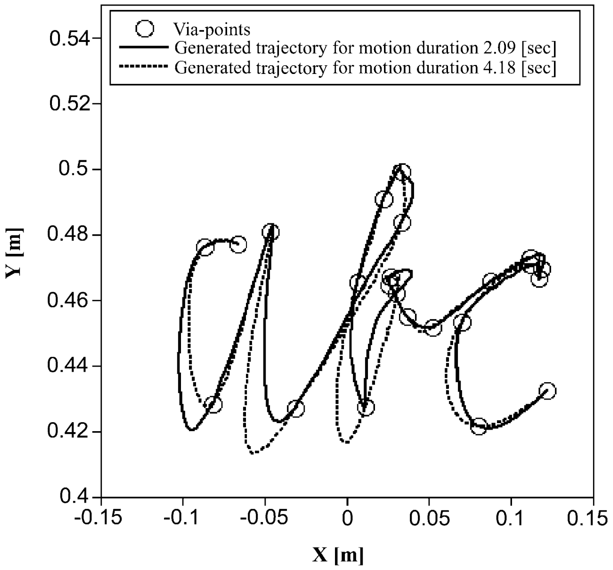

Fig. 12. Handwritten characters ‘abc’ for a motion duration of 2.09 s using the via-point spatial information of a movement with a motion duration of 4.18 s. The trajectories for 2.09 s and for 4.18 s are generated after 35 iterations.

planned the complex trajectory even if the intended motion duration were somewhat different than the learned motion duration.

Models have been developed in which complex sequential movements are generated with motor control synergies or groups of muscles working which overlap in time. An inevitable problem encountered with such models is how to generate complex sequential overlapping movements with proper timing, that is to say, how to determine appropriate temporal information. Researchers in this area have not yet been able to propose an algorithm that can determine overlap timing using only motion duration (Morasso & Sanguineti, 1993; Plamondon & Guerfali, 1998; Schomaker, Thomassen, & Teulings, 1989). Via-points could conceivably be used for switching timing in motor control for sequential movement generation. Recently, Grossberg and Paine (2000) proposed a memory-based model which alters the timings stored in a cortical working memory and which learns through the imitation of handwriting movements. They demonstrated that the model could generate a movement at different speeds other than the learned speed. However, our proposed algorithm is the first to estimate switching timing using only the entire motion duration based on the smoothness criterion.

For robot arm control, our proposed algorithm can regenerate the trajectory from the current state to the final state according to feedback information and can adjust the trajectory. For example, if the current state departs from the planned trajectory significantly, the algorithm can re-plan the trajectory using the via-points from the current state to the final point according to the remaining time.

In the present paper, we have not discussed the adequacy of the revised FIRM incorporating the new algorithm as a human motor planning and control model. In the future, we intend to investigate the adequacy of the proposed model compared to the Central Nervous System by considering the Isogony Principle, the two-thirds power law that relates velocity and curvature, and various human experimental data. However, we have demonstrated that the proposed algorithm can plan almost the same trajectory as that performed by a human. In complex trajectory formation such as handwriting or sign language particularly, our algorithm can generate a trajectory with an intelligible result by using a set of via-points consisting of just spatial information. The experimental results for our proposed algorithm suggest that the CNS might plan a complex trajectory using via-point information and that it might be able to do so using only the spatial information.

We have previously proposed a handwriting model that recognizes characters, which is based on the motor theory of speech perception (Kawato, 1989; Liberman & Mattingly, 1985) in Wada, Koike, Vatikiotis-Bateson, & Kawato, (1995). The via-point obtained in that model is thought to be a feature of the pattern space. However, the via-point is not necessarily a universal feature. Estimating a universal feature by applying the proposed algorithm to trajectory generation and via-point estimation is an important problem to be investigated in future studies.

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Appendix A

We show that the minimum commanded torque change criterion decreases when the motion duration is lengthened. Let us briefly explain this by using the dynamic equation that contains inertia I and viscosity B.

$$\tau(t) = I\ddot{\theta}(t) + B\dot{\theta}(t)$$

Here, the following is considered:

$$t = \alpha s \qquad 0 \le t \le \alpha \ \ (0 \le s \le 1)$$

$$\theta(t) = \theta(\alpha s) = \tilde{\theta}(s)$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta(t) = \frac{\mathrm{d}}{\mathrm{d}s}\theta(\alpha s)\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{\alpha}\frac{\mathrm{d}}{\mathrm{d}s}\tilde{\theta}(s)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\theta(t) = \frac{1}{\alpha^2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}\tilde{\theta}(s)$$

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\theta(t) = \frac{1}{\alpha^3}\frac{\mathrm{d}^3}{\mathrm{d}s^3}\tilde{\theta}(s)$$

The following equation holds:

$$\int_0^{\alpha}\left\{\frac{\mathrm{d}\tau(t)}{\mathrm{d}t}\right\}^2\mathrm{d}t=\int_0^1\left\{I\frac{1}{\alpha^3}\frac{\mathrm{d}^3\tilde{\theta}(s)}{\mathrm{d}s^3}+B\frac{1}{\alpha^2}\frac{\mathrm{d}^2\tilde{\theta}(s)}{\mathrm{d}s^2}\right\}^2\alpha\,\mathrm{d}s$$

$$=\int_0^1\left[\frac{1}{\alpha^5}\left\{I\frac{\mathrm{d}^3\tilde{\theta}(s)}{\mathrm{d}s^3}\right\}^2+\frac{1}{\alpha^3}\left\{B\frac{\mathrm{d}^2\tilde{\theta}(s)}{\mathrm{d}s^2}\right\}^2+\frac{2}{\alpha^4}\left\{I\frac{\mathrm{d}^3\tilde{\theta}(s)}{\mathrm{d}s^3}\right\}\left\{B\frac{\mathrm{d}^2\tilde{\theta}(s)}{\mathrm{d}s^2}\right\}\right]\mathrm{d}s$$

From the equation, it is clear that when the motion duration is short, the time integral of the square of the commanded torque change rate becomes large. Conversely, when the motion duration is long, the time integral becomes small.

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